# BEHAVIOR OF RADIALLY SYMMETRIC SOLUTIONS OF A SYSTEM RELATED TO CHEMOTAXIS

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# BEHAVIOR OF RADIALLY SYMMETRIC SOLUTIONS OF A SYSTEM RELATED TO CHEMOTAXIS

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## 1. Introduction

We consider time-global existence and blow-up of solutions of the following system related to chemotaxis

$$\begin{cases} b_t = \nabla \cdot (\nabla b - \chi b \nabla \phi(s)) & \text{in} \quad \Omega \times (0,\infty), \\ 0 = \Delta s - s + b & \text{in} \quad \Omega \times (0,\infty), \end{cases} \tag{1.1}$$

under the conditions

$$\begin{cases} \dfrac{\partial b}{\partial n} = \dfrac{\partial s}{\partial n} = 0 \quad \text{on} \quad \partial\Omega \times (0,\infty), \\ b(\cdot,0) = b_0 \qquad \text{in} \quad \Omega, \end{cases} \tag{1.2}$$

where $\chi$ is a postive constant and $\phi$ is a smooth function on $(0,\infty)$ with $\phi' > 0$. The system is a simplified Keller-Segel model. Keller-Segel model was introduced by Keller and Segel [11] to describe the initiation of chemotactic aggregation of cellular slime molds. On Keller-Segel model and simplified Keller-Segel models, time-local existence of the solutions has been studied by [19] and blow-up of the solutions has been studied by [4, 10, 9, 14, 18].

The domain $\Omega$ and the non-trivial initial function $b_0$ are only confined to the following case:

(A1) $\Omega$ is the open ball of radius $L$ with center at the origin in $\mathbf{R}^N$ .

(A2) $b_0$ is smooth and nonnegative on $\bar{\Omega}$ , and is radially symmetric when $N \geq 2$ .

Under these assumptions, there exists a unique solution $(b(x,t), s(x,t))$ to (1.1) and (1.2) defined maximal interval of existence $[0, T_{max})$ , which is radially symmetric in $x$ when $N \geq 2$ , smooth in $\bar{\Omega} \times (0, T_{max})$ and $b(x,t) > 0$, $s(x,t) > 0$ for $(x,t) \in \Omega \times (0, T_{max})$ . If $T_{max} < \infty$ ,

$$\limsup_{t \to T_{max}} (\|b(\cdot,t)\|_{L^\infty} + \|s(\cdot,t)\|_{L^\infty}) = \infty,$$

by which we mean that $(b(x,t), s(x,t))$ blows up in finite time.

**Theorem 1** *Let $N = 1$ and $\phi$ be smooth on $(0,\infty)$. Then the solution $(b,s)$ to (1.1), (1.2) is globally bounded, that is, $T_{max} = \infty$ and $(b,s)$ satisfies*

$$\sup_{t\geq 0} (\|b(\cdot,t)\|_{L^\infty} + \|s(\cdot,t)\|_{L^\infty}) < \infty.$$

We put

$$M_a(t) = \int_\Omega b(x,t)|x|^a dx \qquad \text{for } 0 \leq t < T_{max},$$

where $a$ is a positive constant. That is called the moment of order $a$, of $b(\cdot,t)$.

**Theorem 2** *Assume $\phi(s) = s^p$ $(p > 0)$, (A1) and (A2).*

(1) $N = 2$ :

(a) *If $0 < p < 1$, then the solution is globally bounded in time.*

(b) $p = 1$ :

(i) *If $\|b_0\|_{L^1} < 8\pi/\chi$, then the solution is globally bounded in time.*

(ii) *If $\|b_0\|_{L^1} > 8\pi/\chi$ and $M_2(0)$ is sufficiently small, then the solution blows up in finite time.*

(c) *If $p > 1$ and $M_2(0)$ is sufficiently small, then the solution blows up in finite time.*

(2) *If $N \geq 3$ and $M_{(N-2)p+2}(0)$ is sufficiently small, then the solution blows up in finite time.*

**Theorem 3** *Assume $\phi(s) = \log s$, (A1) and (A2).*

(1) *If $N = 2$, then the solution is globally bounded in time.*

(2) $N \geq 3$ :

(a) *If $\chi < 2/(N-2)$, then the solution is globally bounded in time.*

(b) *If $\chi > 2N/(N-2)$ and $M_2(0)$ is sufficiently small, then the solution blows up in finite time.*

## 2. Time-global existence and boundedness

The purpose in this section is to sketch the proofs of Theorem 1 and (i) in Theorems 2 and 3.

Let $G$ be the Green function of $-\Delta + 1$ in $\Omega$ with homogeneous Neumann boundary conditions. For $N \geq 2$ we put

$$E(r) = (2\pi)^{-N/2} r^{(2-N)/2} \kappa_{(N-2)/2}(r) \qquad \text{for } r > 0,$$

where $\kappa_\nu$ is the modified Bessel function of the second kind of order $\nu$ (see [13]). $E$ is a fundamental solution of $-\Delta + 1$.

For the solution $(b,s)$ to (1.1), (1.2) define the functions $S$ and $B$ by

$$S(r,t)=\int_{|x|\le r}s(x,t)dx,\quad B(r,t)=\int_{|x|\le r}b(x,t)dx \tag{2.1}$$

for $0\le r\le L$ and $0\le t<T_{max}$, respectively. $B$ and $S$ satisfy

$$\frac{\partial B}{\partial t} = r^{N-1}\frac{\partial}{\partial r}\left(r^{1-N}\frac{\partial B}{\partial r}\right)+\frac{\chi}{\omega_N}(B-S)\phi'(s)r^{1-N}\frac{\partial B}{\partial r}, \tag{2.2}$$

$$0 = r^{N-1}\frac{\partial}{\partial r}\left(r^{1-N}\frac{\partial S}{\partial r}\right)-S+B, \tag{2.3}$$

for $0<r<L$ and $0<t<T_{max}$, and

$$B(0,t)=S(0,t)=0,\quad B(L,t)=S(L,t)=\|b_0\|_{L^1},$$

where $\omega_N$ is the surface area of the unit sphere $S^{N-1}$ in $\mathbf{R}^N$.

In order to show the boundedness and time-global existence of solutions $(b,s)$ to (1.1), (1.2) , we begin with the following lemmas. These lemmas are shown by the arguments similar to those in [14] and [18], respectively, so we omit the proofs. In what follows, $C$ denotes a generic positive constant depending on $L$ and $N$.

**Lemma 2. 1** *Let $N\ge 2$. Then*

$$s(x,t)\ge C\|b_0\|_{L^1}\quad \text{for } x\in\bar{\Omega} \text{ and } t\in(0,T_{max}).$$

**Lemma 2. 2** *If the following condition*

$$\sup_{0\le t<T_{max}}\|s(\cdot,t)\|_{L^\infty}<\infty,\qquad \sup_{0\le t<T_{max}}\|\nabla\phi(s(\cdot,t))\|_{L^\infty}<\infty,$$

*holds, then $T_{max}=\infty$ and*

$$\sup_{t>0}\|b(\cdot,t)\|_{L^\infty}<\infty.$$

For the following lemma, see [18].

**Lemma 2. 3** *Let $N\ge 2$. Then the following holds :*

$$B(|x|,t)E(|x|)\le s(x,t)\le C\|b_0\|_{L^1}E(|x|)\quad \text{in } \Omega\setminus\{0\}\times(0,T_{max}).$$

**Sketch of proofs of Theorem 1 and (i) in Theorems 2 and 3 .** By Lemmas 2. 1 , 2. 2 and Appendixes in [14] and [18], it suffices to show that

$$\sup_{0\le t<T_{max}}\|s(\cdot,t)\|_{L^\infty}<\infty,\qquad \sup_{0\le t<T_{max}}\|\nabla s(\cdot,t)\|_{L^\infty}<\infty. \tag{2.4}$$

In the case of $N=1$, (2.4) is shown by the arguments similar to those in [14]. Hence we will prove (2.4) in the case of $N\ge 2$.

We put

$$\Phi(u)=\begin{cases} p\|b_0\|_{L^1}^p u^{p-1} & \text{in the case of Theorem 2 ,}\\ u^{-1} & \text{in the case of Theorem 3}\end{cases}$$

for $u > 0$. It follows from Lemma 2. 3 , (2.2) and $\partial B/\partial r \geq 0$ that $B$ satisfies

$$\frac{\partial B}{\partial t} \leq r^{N-1}\frac{\partial}{\partial r}\left(r^{1-N}\frac{\partial B}{\partial r}\right) + \frac{\chi}{\omega_N}\Phi(E)r^{1-N}\frac{\partial B}{\partial r}.$$

We can construct the function $W(r)$ such that

$$W(r) \sim r^N \text{ as } r \to 0,$$

$$\|b_0\|_{L^1} < W(L) \quad \text{and} \quad B(r,0) \leq W(r) \qquad \text{for } 0 \leq r \leq L,$$

and that

$$0 \geq r^{N-1}\frac{d}{dr}\left(r^{1-N}\frac{dW}{dr}\right) + \frac{\chi}{\omega_N}B\phi'(s)r^{1-N}\frac{dW}{dr} \text{ for } 0 < r < L.$$

Hence, the comparison theorem yields that

$$B(r,t) \leq W(r) \qquad \text{for } 0 \leq r \leq L, 0 \leq t < T_{max},$$

which implies $B(r,t) \leq Cr^N$.

Since $B(r,t) \leq Cr^N$ for $0 \leq r \leq L$ and $0 \leq t < T_{max}$, it follows from (2.3) that

$$S(r,t) \leq Cr^N \qquad \text{for } 0 \leq r \leq L, 0 \leq t < T_{max}.$$

Then we have that

$$|\nabla s(x,t)| = \frac{|S(|x|,t) - B(|x|,t)|}{\omega_N |x|^{N-1}} \leq C$$

for $x \in \Omega$ and $0 \leq t < T_{max}$. The boundedness of $\|s(\cdot,t)\|_{L^\infty}$ with respect to $t \in [0, T_{max})$ follows from the estimate above of $|\nabla s|$ and

$$\min_{x \in \bar{\Omega}} s(x,t) \leq \frac{\|b_0\|_{L^1}}{|\Omega|} \qquad \text{for } 0 \leq t < T_{max},$$

where $|\Omega|$ is the volume of $\Omega$. Thus the proofs of (i) of Theorems 2 and 3 are complete.

## 3. Blow-up of solutions

The purpose in this section is to show the blow-up of solutions for the system (1.1), (1.2) in the case of $N \geq 2$ .

In order to show the blow-up of solutions $(b, s)$ to (1.1), (1.2) in [14] and [18], a differential inequality on a moment $M_k(t)$ of $b$ is constructed by use of some estimates of $s$, and under some conditions on $b_0$ it is shown that the moment of $b$ converges to 0 as $t$ tends some $T_0 \in (0, \infty)$ by use of the differential inequality.

The following lemma is an immediate consequence of Hölder's inequality.

**Lemma 3. 1** *Let $f$ be an integrable function on $\Omega$, and $p_1$, $p_2$ and $p_3$ be numbers satisfying $0 \leq p_1 < p_2 < p_3$. Then*

$$\int_\Omega |f||x|^{p_2}dx \leq \left\{\int_\Omega |f||x|^{p_1}dx\right\}^{(p_3-p_2)/(p_3-p_1)} \left\{\int_\Omega |f||x|^{p_3}dx\right\}^{(p_2-p_1)/(p_3-p_1)}.$$

Let $S$ and $B$ be the same functions as in (2.1). The following lemmas are stated in [15] and [18].

**Lemma 3. 2** *The inequality holds :*

$$\begin{aligned} \frac{d}{dt}M_k(t) \;&\le\; k(k+N-2)\int_\Omega b(x,t)|x|^{k-2}dx \\ &+\frac{k\chi}{\omega_N}\int_\Omega \phi'(s(x,t))b(x,t)\left\{S(|x|,t)-B(|x|,t)\right\}|x|^{k-N}dx \end{aligned}$$

*on* $(0,T_{max})$, *where* $k\ge 2$ .

**Lemma 3. 3** *Let* $N\ge 3$. *There exists a positive constant* $\delta$ *such that*

$$\frac{\partial}{\partial r}\left(r^{N-1}s(x,t)\right)\ge 0 \qquad in\ \{x\in R^N : |x|\le\delta\}\times(0,T_{max}),$$

*where* $r=|x|$.

**Lemma 3. 4** *Let* $N\ge 2$. *Then the following holds :*

$$s(x,t)\le\frac{1}{\omega_N|x|^{N-1}}\int_{|y|=|x|}E(|x-y|)d\sigma\|b_0\|_{L^1}+\int_\Omega K(x,y)b(y,t)dy$$

*in* $\Omega\setminus\{0\}\times(0,T_{max})$.

**Sketch of proof of (ii) of Theorem 2 .** Let $k=(N-2)p+2$. In order to prove the theorem, it suffices to show the following inequality

$$\begin{aligned} \frac{d}{dt}M_k(t) \;&\le\; k(k+N-2)\|b_0\|_{L^1}^{2/k}M_k(t)^{(k-2)/k} \\ &+C\|b_0\|_{L^1}^{p+(k-2)/k}M_k(t)^{2/k}-C\|b_0\|_{L^1}^{p+1} \end{aligned} \tag{3.1}$$

for $t\in(0,T_{max})$. In fact, if $M_k(0)$ is sufficiently small so that the right-hand side of (3.1) is negative at $t=0$, there exists $T_0\in(0,\infty)$ such that

$$M_k(t)\to 0 \quad\text{as}\quad t\to T_0.$$

Hence, $T_{max}$ must be finite and $T_{max}\le T_0$. By Appendixes in [14] and [18], we have

$$\limsup_{t\to T_{max}}\|b(\cdot,t)\|_{L^\infty}=\infty.$$

Let us first show (3.1) in the case of $p\ge 1$. Using Lemmas 2. 3 and the properties of the fundamental solution, we obtain that

$$\int_\Omega s^{p-1}(x,t)b(x,t)B(|x|,t)|x|^{k-N}dx \;\ge\; C\|b_0\|_{L^1}^{p+1}. \tag{3.2}$$

and that

$$S(|x|,t)\le C\|b_0\|_{L^1}|x|^2. \tag{3.3}$$

It follows from Lemma 2. 3 and (3.3) and the properties of the fundamental solution that

$$\int_\Omega s^{p-1}(x,t)b(x,t)S(|x|,t)|x|^{k-N}dx \le C\|b_0\|_{L^1}^p M_2(t). \tag{3.4}$$

Lemma 3. 2 together with (3.2), (3.4) and Lemma 3. 1 yields (3.1).

Let us consider the case $0 < p < 1$. By Lemmas 2. 3 and the properties of the fundamental solution, we have

$$\begin{aligned}&\int_\Omega s^{p-1}(x,t)b(x,t)B(|x|,t)|x|^{k-N}dx \\ &\ge C\|b_0\|_{L^1}^{p-1}\int_\Omega b(x,t)B(|x|,t)dx = \frac{C}{2}\|b_0\|_{L^1}^{p+1}.\end{aligned} \tag{3.5}$$

It follows from Lemmas 2. 3 and 3. 3 and the properties of the fundamental solution that

$$\int_{|x|\le\delta} s^{p-1}(x,t)b(x,t)S(|x|,t)|x|^{k-N}dx \le C\|b_0\|_{L^1}^p\int_{|x|\le\delta} b(x,t)|x|^2dx. \tag{3.6}$$

By Lemma 2. 1 and (3.3), we have

$$\begin{aligned}&\int_{\delta\le|x|\le L} s^{p-1}(x,t)b(x,t)S(|x|,t)|x|^{k-N}dx \\ &\le C\|b_0\|_{L^1}^p\int_{\delta\le|x|\le L} b(x,t)|x|^{k-N+2}dx \\ &\le C\|b_0\|_{L^1}^p\delta^{k-N}\int_{\delta\le|x|\le L} b(x,t)|x|^2dx.\end{aligned} \tag{3.7}$$

Combining (3.6) with (3.7) yields that

$$\int_\Omega s^{p-1}(x,t)b(x,t)S(|x|,t)|x|^{k-N}dx \le C\|b_0\|_{L^1}^p M_2(t). \tag{3.8}$$

By (3.5) and (3.8), the similar argument to that in the case of $p \ge 1$ gives us (3.1). Thus the proof is complete.

**Sketch of proof of (ii) in Theorem 3 .** Observe that it follows from Lemma 3. 4 and the properties of the fundermental solution that for $0 \le t < T_{max}$ and $0 < |x| \le L/2$,

$$s(x,t) \;\le\; \left\{\frac{1}{\omega_N(N-2)|x|^{N-2}} + C(|x|^{3-N}+1)\right\}\|b_0\|_{L^1}.$$

For $0 \le t < T_{max}$ and $0 < \delta \le L/2$, we then have that

$$\begin{aligned}&\int_\Omega b(x,t)B(|x|,t)\frac{1}{|x|^{N-2}s(x,t)}dx \\ &\ge\; \frac{1}{\|b_0\|_{L^1}}\left\{\frac{1}{(N-2)\omega_N}+C\delta\right\}^{-1}\int_{|x|\le\delta} b(x,t)B(|x|,t)dx\end{aligned}$$

$$\geq \frac{(N-2)\omega_N}{2(1+C\delta)\|b_0\|_{L^1}} B(\delta,t)^2 \tag{3.9}$$
$$\geq \frac{(N-2)\omega_N}{2(1+C\delta)\|b_0\|_{L^1}} \left(\|b_0\|_{L^1} - \frac{1}{\delta^2} M_2(t)\right)_+^2$$
$$\geq \frac{(N-2)\omega_N}{2(1+C\delta)} \left(\|b_0\|_{L^1} - \frac{2}{\delta^2} M_2(t)\right),$$

where $(\cdot)_+ = \max\{\cdot, 0\}$. It follows from Lemma 3. 3 that

$$\int_\Omega b(x,t) S(|x|,t) \frac{1}{|x|^{N-2} s(x,t)} dx$$
$$= \omega_N \int_{|x|\leq\delta} b(x,t)|x|^2 dx + \int_{\delta\leq|x|\leq L} b(x,t) S(|x|,t) \frac{1}{|x|^{N-2} s(x,t)} dx \tag{3.10}$$
$$\leq CM_2(t)$$

in $(0, T_{max})$. Hence, combining Lemma 3. 2 with (3.9) and (3.10) concludes that

$$\frac{d}{dt} M_2(t) \leq \left\{2N - \frac{(N-2)\chi}{1+C\delta}\right\} \|b_0\|_{L^1} + C\chi(1+\delta^{-2}) M_2(t)$$

on $(0, T_{max})$. Suppose that $\delta$ is sufficiently small so that $2N(1+C\delta) - (N-2)\chi < 0$. Using the argument similar to that in the sketch of proof of Theorem 2 , then we have the proof.

# References


[1] P. Biler and T. Nadzieja, Existence and nonexistence of solutions for a model of gravitational interactions of particles, I, Colloq. Math., **66** (1994), 319–334.

[2] P. Biler, Existence and nonexistence of solutions for a model of gravitational interactions of particles, III, Colloq. Math., **68** (1995), 229–239.

[3] S. Childress, Chemotactic collapse in two dimensions, Lecture Notes in Biomath., vol. 55, Springer, Berlin-Heidelberg-New York, 1984, 61–66.

[4] S. Childress and J. K. Percus, Nonlinear aspects of chemotaxis, Math. Biosci., **56** (1981), 217–237.

[5] J. I. Diaz and T. Nagai, Symmetrization in a parabolic-elliptic system related to chemotaxis, Adv. Math. Sci. Appl., **5** (1995), 659–680.

[6] G. B. Folland, Introduction to Partial Differential Equations, Princeton University Press, Princeton, New Jersey, 1976.

[7] P. R. Garabedian, Partial Differential Equations, John Wiley & Sons, New York, 1964.

[8] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Math., vol. 840, Springer, Berlin-Heidelberg-New York, 1981.

[9] M. A. Herrero and J. J. L. Velázquez, Singularity patterns in a chemotaxis model, Math. Ann., to appear.

[10] W. Jäger and S. Luckhaus, On explosions of solutions to a system of partial differential equations modelling chemotaxis, Trans. Amer. Math. Soc., **329** (1992), 819–824.

[11] E. F. Keller and L. A. Segel, Initiation of slime mold aggregation viewed as an instability, J. Theor. Biol., **26** (1970), 399–415.

[12] I. R. Lapidus and M. Levandowsky, Modeling chemosensory responses of swimming eukarvotes, Biological Growth and Spread, Proceedings, Heidelberg 1979, Lecture Notes in Biomathematics, vol. 38, Springer-Verlag, 1980.

[13] T. Myint-U and L. Debnath, Partial Differential Equations for Scientists and Engineers, North-Holland, New York, 1987.

[14] T. Nagai, Blow-up of radially symmetric solutions to a chemotaxis system, Adv. Math. Sci. Appl., **5** (1995), 581–601.

[15] T. Nagai and T. Senba, Global existence and blow-up of radially symmetric solutions to a parabolic-elliptic system to chemotaxis, preprint.

[16] V. Nanjundiah, Chemotaxis, signal relaying, and aggregation morphology, J. Theor. Biol., **42** (1973), 63–105.

[17] R. Schaaf, Stationary solutions of chemotaxis systems, Trans. Amer. Math. Soc., **292** (1985), 531-556.

[18] T. Senba, Blow-up of radially symmetric solutions to some systems of partial differential equations modelling chemotaxis, Adv. Math. Sci. Appl., to appear.

[19] A. Yagi, Norm behavior of solutions to the parabolic system of chemotaxis, preprint.